

Singular Scaling Functions in Clustering Phenomena

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We study clustering in a stochastic system of particles sliding down a fluctuating surface in one and two dimensions. In steady state, the density-density correlation function is a scaling function of separation and system size. This scaling function is singular for small argument — it exhibits a cusp singularity for particles with mutual exclusion, and a divergence for noninteracting particles. The steady state is characterized by giant fluctuations which do not damp down in the thermodynamic limit. The autocorrelation function is a singular scaling function of time and system size. The scaling properties are surprisingly similar to those for particles moving in a quenched disordered environment that results if the surface is frozen.

PACS numbers: 05.40.-a, 47.40.-x, 64.75.+g

I. INTRODUCTION

Clustering phenomena are widespread. On the one hand, they include the growth of regions of a homogeneous ordered phase, such as droplets of a liquid or domains in a magnet; on the other hand, they also include the formation of inhomogeneous structures such as those that arise in granular gases. In general, clusters break and re-form due to the action of noise, whether thermal or induced by external driving. Concomitantly, the distribution of clusters — their sizes and spatial locations — keep evolving in time. A useful diagnostic of the nature of clustering, both in steady state and during the approach to it, is provided by the two-point density-density correlation function. In this paper, we will discuss the scaling properties of correlation functions, and show that clustered states of different sorts correspond to different types of singularities in the corresponding scaling functions.

One of the most familiar and well studied examples of clustering is afforded by phase ordering kinetics. Here the clusters are simply domains of ordered phases which form following a rapid quench from an initially disordered state to a regime which favours ordering. The coarsening of ordered domains in time is captured by the two-point correlation function; it shows simple scaling properties when spatial separations are scaled by the mean domain size, itself a growing function of time [1]. In steady state, the system size replaces the domain size in the scaling function.

There are a number of situations in which particles cluster in such a way that the resulting density profile is patchy and far from homogeneous. For instance, particles may arrange themselves in a critical state with power law decays of the correlation function, e.g. molecules in a fluid at the critical point [2] or galaxies in the universe [3]. However, the sort of clustering of interest in this paper is different in that it is associated with long range order, unlike a critical state, yet has strong fluctuating properties which make it quite different from a conventional phase ordering system.

We focus on particles subjected to randomly fluctu-

ating forces, which result in an inhomogeneously clustered state. Particles advected by fluid flows exhibit a clustering tendency whose strength depends on the compressibility of the fluid and the inertia of the particles [4, 5, 6]. Earlier studies have shown that in the absence of noise, particle trajectories can coalesce as time advances [7, 8]. Not surprisingly, external noise — which allows coalesced clusters to break up — has a profound effect, and the dynamical steady state which results from the balance between making and breaking clusters has interesting characteristics. Our aim in this paper is to discuss the nature of this state, with a focus on the scaling properties of correlation functions. We study simple stochastic models of particles sliding down a fluctuating surface [9, 10, 11, 12, 13, 14]. As for the case of phase ordering kinetics, the two-point correlation function has simple scaling properties. However, now the associated scaling function develops singularities — these are *cusp singularities* for particles with mutual exclusion, and *divergences* for noninteracting particles. The occurrence of such singularities is quite robust, and survives as parameters of the model and the dimension are changed. Moreover, recent work has shown that such singularities are found also in other completely different systems which exhibit inhomogeneous clustering [15, 16].

In a finite system, a dynamical steady state is reached in which correlations are scaling functions of the separation and system size. Like conventional phase ordering systems, this system exhibits long range order, but unlike them, the scaling functions are singular at small argument, indicative of a steady state which is intrinsically different from a phase separated state in an equilibrium system. A crucial point about the steady state is the existence of strong fluctuations which do not die down in the thermodynamic limit, due to which the degree of ordering in the system keeps fluctuating, but never vanishes.

II. SCALING IN PHASE ORDERING SYSTEMS

Phase ordering kinetics deals with the onset of order in a system which is brought rapidly from a disordered state to an order-promoting regime [1]. Domains of competing phases coarsen in time, a process which is captured by the two-point correlation function $C(r, t) = \langle \sigma_i(t) \sigma_{i+r}(t) \rangle$ where $\sigma_i = \pm 1$ is an Ising variable which describes a spin in a magnet, or an occupancy variable $(1 + \sigma_i)/2$ within a lattice gas description. In an infinite system, the correlation function has the scaling form

$$C_S(r, t) = Y(r/\mathcal{L}(t)) \quad (1)$$

in the scaling limit $r \rightarrow \infty$, $t \rightarrow \infty$ with the ratio $y = r/\mathcal{L}(t)$ held fixed. Here $\mathcal{L}(t)$ is a growing, time-dependent length scale which describes the typical linear size of a cluster at time t ; typically it grows as a power law in time, $\mathcal{L}(t) \sim t^{\beta'}$.

The scaling form of Eq. (1) holds only if the separation $r \gg$ the correlation length ξ . In general, the correlation function $C(r, t)$ contains both the scaling part $C_S(r, t)$ and an analytic part $C_A(r)$ which comes from short-distance ($r \ll \xi$) correlations, independent of $\mathcal{L}(t)$. In practice, it is important to account for the occurrence of $C_A(r)$ while analyzing data for the correlation function to see whether scaling holds.

When the system size L is infinite, the domain size $\mathcal{L}(t)$ can grow indefinitely, with the scaling form of Eq. 1 continuing to hold. However, if L is finite, then Eq. 1 ceases to hold once t is large enough that $\mathcal{L}(t) \approx L$. Beyond this, the system reaches steady state and Eq. (1) is replaced by

$$\bar{C}_S(r, L) = \bar{Y}(r/L). \quad (2)$$

There are two important characteristics of the scaling functions $Y(y)$ of Eq. 1 and $\bar{Y}(y)$ of Eq. 2: (i) the scaling function has a finite intercept, (ii) the scaling function falls linearly for small y . Property (i) is important as the intercept is a measure of long range order (LRO). To see this for $Y(y)$, recall that LRO is defined in an infinite system in steady state as $m_0^2 = \lim_{r \rightarrow \infty} \langle \sigma_i \sigma_{i+r} \rangle$. Now consider two points separated by an arbitrarily large but fixed distance r in a coarsening system. As $t \rightarrow \infty$, domain sizes grow without bound and the interior of each domain is well approximated by steady state. Thus at large times, both points would belong to the same domain with probability one, implying that their correlation equals m_0^2 . Since $t \rightarrow \infty$ leads to $y \rightarrow 0$, no matter how large r is, this implies that the intercept in Fig. 1 is m_0^2 .

Further, property (ii) which describes the manner in which the scaling function falls from the intercept value m_0^2 is also significant. For scalar order parameters, it falls linearly: $Y(y) \approx m_0^2 - b|y|$ as $y \rightarrow 0$ [1]. For a d -dimensional system, this translates into a decay $\sim [k\mathcal{L}(t)]^{-(d+1)}$ for the structure factor at large wave-vector k . This power law decay is known as the Porod Law [17].

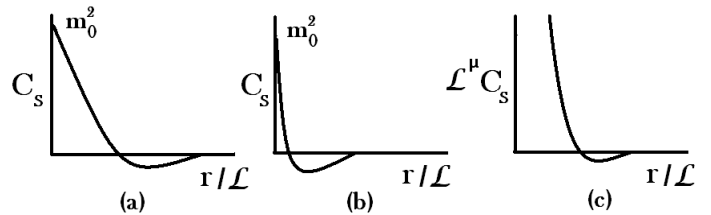


FIG. 1: The behaviour of the scaling part of the correlation function is depicted for (a) normal phase ordering systems (b) a system of particles with exclusion, sliding down a fluctuating surface (Model E) (c) noninteracting particles sliding down a fluctuating surface (Model F). The linear drop of C_S in (a) [the Porod law] is replaced by a cusp singularity in (b), and by a divergence in (c).

The linear fall of correlations for $|y| = |r/\mathcal{L}| \ll 1$ can be traced to the existence of well-defined interfaces between two phases: Two points a distance r apart are positively correlated if both belong to the same domain, and negatively correlated if a single interface intersects the line joining them; for small values of r/\mathcal{L} , the chance of this happening is proportional to r and inversely proportional to \mathcal{L} , accounting for the linear drop of the scaling function at small argument.

In subsequent sections we investigate how far these properties are respected when we have clustering of an inhomogeneous sort. We find that there is a class of inhomogeneous clustering phenomena, exemplified by models of particles sliding on randomly fluctuating surfaces, for which most of the properties discussed above are valid, except for the form of singular behaviour of the scaling function at small argument. These models and a discussion of the singularities they display are discussed in the following sections.

III. MODEL: PARTICLES SLIDING ON A FLUCTUATING SURFACE

The model consists of a system of overdamped particles sliding down under gravity along the local slope of a fluctuating interface (Fig. 2) [9, 12]. In 1-d, we model the interface through the single-step model [18]. It consists of a series of links, with the slope of the link between the i 'th and $(i+1)$ 'th site being $\tau_{i+\frac{1}{2}} = \pm 1$, so that a typical $\{\tau\}$ configuration is $//\backslash\backslash\backslash\backslash\backslash\backslash\backslash\backslash$. The height at site i is given by $h_i = \sum_{j=1}^i \tau_{j-\frac{1}{2}}$. The elementary dynamical move is a stochastic corner flip, which involves the exchange of adjacent τ 's: the transition $\backslash \rightarrow /$ occurs with rate p_1 , while the reverse transition $/ \rightarrow \backslash$ occurs with rate q_1 . For large r and t , with $p_1 = q_1 = 1$ the interface dynamics reduces to the Edwards-Wilkinson model [19], while with $p_1 \neq q_1$, it reduces to the Kardar-Parisi-Zhang (KPZ) model [20]. Note that the overall slope of the interface $\mathcal{T} = \sum_{i=1}^L \tau_{i+\frac{1}{2}}$, is conserved; we use peri-

odic boundary conditions and work with interfaces with no overall tilt i.e. $\mathcal{T} = 0$.

For the dynamics of the particles, there are two cases: particles with mutual exclusion (E) [9], and particles with no interactions, i.e. free (F) [12].

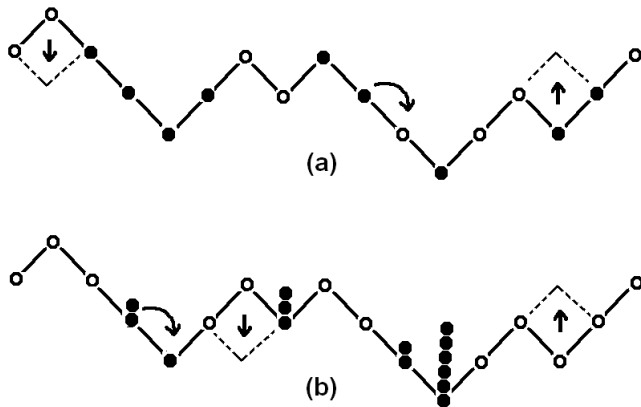


FIG. 2: Typical configurations and moves for lattice models of particles sliding down a fluctuating surface in 1-d. (a) corresponds to Model E (particles with exclusion) and (b) to Model F (particles with no interactions).

In case E (Fig. 2a), a particle and hole at sites i and $i + 1$ exchange at a rate which is governed by the slope variable $\tau_{i+\frac{1}{2}}$. A move down the slope ($\bullet \backslash o \rightarrow o \bullet$ or $o / \bullet \rightarrow \bullet / o$) occurs with rate p_2 , while the reverse transitions occur with rate $q_2 < p_2$.

In case F (Fig. 2b), there is no limit on the occupancy of any site, and a particle moves to its neighbouring site with rate p_2 or q_2 depending on whether it traverses the intervening τ bond in the downward or upward direction. If $q_2 = 0$, particles move only downward and coalesce at the bottoms of valleys. Nevertheless, particle clusters can break again if they are carried, by surface evolution, to a local hilltop; subsequently, particles are equally likely to fall left or right, leading to a break-up of the cluster.

Notice that in both cases E and F, the particles are passive scalars: while the evolution of particles occupancies $\{n_i\}$ is governed by the interface configuration $\{\tau_{i+\frac{1}{2}}\}$, the evolution of the interface is autonomous and is not influenced by the particles.

Model E with EW dynamics is related to the Lahiri-Ramaswamy (LR) model of sedimenting colloidal crystals [21, 22]. In the present context, the general LR model allows for the particles to act back on the fluctuating surface, in which case phase-separated and disordered phases ensue [22, 23]. The case of passive particles under consideration lies at the boundary between these more conventional phases [24].

A lattice model closely related to model F with KPZ dynamics was studied by Drossel and Kadar [25, 26]; their results are discussed in Section 5 along with ours. Interestingly, model F with KPZ dynamics relates to the problem of passive scalar advection by a noisy Burgers

flow. This follows from the well known mapping [20] between the KPZ equation and the noisy Burgers equation which describes a compressible flow, and from the observation that movement of a particle along the local slope in the present context translates, through the mapping, to a particle being advected by the local velocity field.

IV. SLIDING PARTICLES WITH EXCLUSION

We turn to the properties of a system of particles with mutual exclusion on a fluctuating interface, evolving under the dynamics discussed in the previous section.

A. Coarsening

At $t = 0$, we start out with a configuration of the interface drawn from steady state, and particles distributed randomly on sites. Numerical simulations of this process using a very large system size were performed to monitor the two-point correlation functions $C(r, t)$, averaged over many histories [9, 10]. Particles and surface elements were updated at equal rates. The results show that there is a good scaling collapse of the data to the form of Eq. 1, with $\mathcal{L}(t) \sim t^{1/z}$ where z is the dynamical exponent which characterizes the interface dynamics. This is because in time t , new valleys of size $t^{1/z}$ form, so that particles in basins of that size are drawn together. In 1 dimension, we have $z = 2$ for EW evolution, and $z = 3/2$ for KPZ evolution.

The significant new point is that the scaling function has a cusp singularity

$$C_S(y) \approx m_0^2 - b|y|^\alpha \quad \text{as } y \rightarrow 0 \quad (3)$$

with $y = r/\mathcal{L}(t)$. This translates into a tail $\sim (k\mathcal{L}(t))^{1+\alpha}$ for the wave-vector dependence of the structure factor, implying non-Porod behaviour. The cusp exponent α is found to be $\simeq 0.5$ with EW evolution and $\simeq 0.25$ with KPZ evolution of the interface [10]. The value of α is found to be unaffected by variations of the density of particles or the relative ratio of interface and particle update rates. Moreover, the phenomenon persists with 2-d interfaces as well [11]; with KPZ dynamics, it is found that $\alpha \simeq 0.38$ (see Section 4.4).

B. Steady state and Cluster size distribution

A study of finite systems of size L shows that in steady state, the correlation function is a scaling function of separation and size, as in Eq. 2 [10]. Once again, the scaling function shows a cusp singularity, and the value of the cusp exponent α matches that obtained in the coarsening studies discussed in Section 4.1.

A clue to the nature of the state comes from a study of the distribution $P(\ell)$ of cluster sizes ℓ [10]. A cluster is defined as a stretch of continuously occupied sites,

with unoccupied perimeter sites. In the 1-d EW case, it is found that $P(\ell) \sim \ell^{-\theta}$ with $\theta \simeq 1.8$; symmetry implies the same form for hole clusters. In the 1-d KPZ case, simulations reveal that hole clusters follow a power law decay with $\theta \simeq 1.85$, whereas the decay for particle clusters do not seem to follow a simple power law.

The power-law decay of $P(\ell)$ can be related to the cusp observed in the two-point correlation function [10] within the independent interval approximation (IIA) which is based on the premise that the occurrence of successive intervals (clusters of particles and holes) are independent events. With this approximation, the Laplace transforms $\tilde{P}(s) = \int_0^\infty d\ell e^{-\ell s} P(\ell)$ and $\tilde{C}(s) = \int_0^\infty dr C(r)$ are related by [30]

$$s \left[1 - s\tilde{C}(s) \right] = \frac{2}{\langle \ell \rangle} \frac{1 - \tilde{P}(s)}{1 + \tilde{P}(s)} \quad (4)$$

where $\langle \ell \rangle$ is the mean cluster size. For the slow power law decay $P(\ell) \sim \ell^{-\theta}$, where ℓ is limited by $\ell = L$, we have $\langle \ell \rangle \simeq aL^{2-\theta}$ and $\tilde{P}(s) \approx 1 - bs^{\theta-1}$ for $1/L \ll s \ll 1$. Equation 4 then leads to the scaling from of Eqs. 2 and 3, with the cusp exponent given by

$$\alpha = 2 - \theta. \quad (5)$$

The occurrence of LRO in this fluctuating system can also be understood by noting that extremal statistics applied to $P(\ell) \sim \ell^{-\theta}$ with $1 < \theta < 2$ leads to the conclusion that the largest clusters are of size $\sim L$.

C. Adiabatic limit: Coarse-grained depth (CD) model

We can calculate θ and thus α within the IIA for a class of models motivated by considering the adiabatic limit of infinitely slow interface evolution. Then the height field $\{h_i\}$ is a quenched random variable with $h_i = \sum_{j=1}^i \tau_{j-\frac{1}{2}}$. The steady state of the particle distribution is a thermal equilibrium state with temperature defined by $T = V \ln(p_2/q_2)$ where V is the potential energy drop across a single link. In the limit of low T , this equilibrium state is characterized by a filling level akin to the Fermi level, namely the height up to which the particles are filled. In this limit, the correlation function $\langle s_i s_{i+r} \rangle$ coincides with $C(r)$.

This motivates us to define and study more general coarse-grained depth models (CD models) [9] with variable filling levels

$$s_i(t) = \text{sgn} [h_i(t) - \bar{h}] \quad (6)$$

where \bar{h} is the height of a surface cut. Another natural choice for \bar{h} is the instantaneous average height of the configuration $\{h_i\}$, but with this definition the cut need not coincide with the filling level defined in the previous paragraph. Yet another possibility for the cut is to pin it to the instantaneous height of the first site. For the 1-d

EW and KPZ interfaces, the configuration of the interface is given by the trajectory of a random walk and the filling level defines a cut of this trajectory. The intervals between successive crossings then follow a distribution $p(\ell) \sim \ell^{-3/2}$. Using Eq. 5, this leads to a cusp exponent $\alpha = 1/2$.

D. Higher dimensions

Singularities in scaling functions are also found to occur in a system of particles sliding down a fluctuating rough surface in two dimensions [11]. The surface is modeled by a discrete solid-on-solid model with a simple growth rule: a site on a square lattice is selected a random and its height is decreased by 2 units, provided the height at all four of its neighbouring sites is lower. The asymptotic properties of this model are expected to be the same as those of the (2+1)-dimensional KPZ equation. Particles are initially distributed at random, and updated by choosing a random particle, attempting to move it to a randomly chosen neighbour, and actually moving it provided the local slope is downward and the target site is unoccupied.

Results on coarsening [11] indicate that Eq. 1 holds, with $\mathcal{L}(t) \sim t^{1/z}$ and $z \simeq 1.6$, close to the value of the dynamical exponent which defines the relaxation of a (2+1)-dimensional KPZ surface. The cusp exponent α characterizing the decay of the scaling form of the density-density correlation function is $\simeq 0.38$. A study of the CD model (with \bar{h} taken to be the average height) shows that Eq. 2 holds, with $\alpha \simeq 0.43$.

In steady state, we may relate the cusp exponent α in the CD model to first return probabilities through the following argument [11]. Consider a self-affine surface with roughness exponent χ , and define $P(\ell)$ as the probability that the surface first returns to its starting height a distance ℓ along an arbitrary linear direction. $P(\ell)$ decays as power law for $\ell \ll L$: $P(\ell) \sim \ell^{-(2-\chi)}$. On applying the IIA to different segments on a straight line connecting two points in a distance r apart, Eq. 5 leads to the conclusion that $\alpha = \chi$, i.e. the cusp exponent for the CD model is determined by the roughness exponent of the fluctuating, driving surface. This is borne out by the results in both 1 and 2 dimensions ($\alpha = \chi = 1/2$ in 1-d; $\alpha \simeq 0.43, \chi \simeq 0.4$ in 2-d).

E. Dynamics

As with static (equal-time) correlation functions in steady state, dynamical properties also exhibit scaling characterized by singular scaling functions [14]. The autocorrelation function $A(t) = \langle \sigma_i(0) \sigma_i(t) \rangle$ was monitored by Monte Carlo simulation, and found to be described by a scaling function $W(t/L^z)$, in the scaling limit $t \rightarrow \infty, L \rightarrow \infty$ with t/L^z held finite. Parallel to the static situation, the scaling function $W_S(w)$ exhibits

a cusp singularity at small argument:

$$W_S(w) \approx m_0^2(1 - b'|w|^\beta) \quad w \rightarrow 0 \quad (7)$$

where $w = t/L^z$, m_0^2 measures long-range order and β is the cusp exponent. The results of simulations indicate that $\beta \simeq 0.22$ for Edwards-Wilkinson surface dynamics and $\beta \simeq 0.18$ for KPZ surface dynamics [14].

Just as for static correlations (section 4.3), insight into singular scaling for dynamic correlations can be obtained from analytic calculations on the corresponding CD model. To this end, the autocorrelation function $\langle s_i(0)s_i(t) \rangle$ of the CD model with EW evolution was calculated [14] and shown to follow the form of Eq. 2 with $m_0 = 1$ and $\beta = 0.25$.

Further, aging functions of the form $\mathcal{A}(t_1, t_2) \equiv \langle \sigma_i(t_1)\sigma_i(t_2) \rangle$ were investigated for a coarsening system, starting from a state with randomly distributed particles [14]. If $t_1, t_2 \gg 1$, the aging function \mathcal{A} is a function of the ratio t_1/t_2 alone. For $t_1 \gg t_2$, we find $\mathcal{A} \approx m_0^2[1 - b_1(t_2/t_1)^\beta]$, whereas for $t_1 \ll t_2$, the function \mathcal{A} follows a power law decay $(t_1/t_2)^\gamma$. Simulations show that $\gamma \simeq 0.69$ for EW interface dynamics, $\gamma \simeq 0.82$ for KPZ dynamics, and $\gamma \simeq 0.82$ for the CD model.

F. Ordering and Giant fluctuations

As we have seen, the two-point correlation function characterizing the steady state approached by a system of particles driven by a fluctuating surface exhibits long range order. Let us ask for a *one-point* function or order parameter which can be used to characterize the steady state. To this end, we monitored the Fourier transform

$$Q(k) = \left| \frac{1}{L} \sum_{j=1}^L e^{ikj} n_j \right|, \quad k = \frac{2\pi m}{L}, \quad (8)$$

where $m = 1, 2, \dots, L-1$. The expectation value $Q_1^* = \langle Q_1 \rangle$ of the longest-wavelength mode $Q_1 = Q(2\pi/L)$ corresponding to $m = 1$ is a putative order parameter. For a state which exhibits complete phase separation in a half-filled system, $Q_1^* \simeq 0.32$, while for a disordered state, $Q_1^* = 0$; however, as we will see below, $Q_1^* = 0$ does not necessarily imply a disordered state. Figure 3 shows schematically the behaviour of $\langle Q(k) \rangle$ as a function of k for various values of L . For fixed wave-vector $k \neq 0$, we see that $\langle Q(k) \rangle \rightarrow 0$ as $L \rightarrow \infty$. To study the $k \rightarrow 0$ limit, we fix the mode number m (e.g. $m = 1$ or $m = 2$) and monitor $Q_m^* = \langle Q_m \rangle \equiv \langle Q(2\pi m/L) \rangle$. We see that Q_1^*, Q_2^* etc. approach a finite limit as $L \rightarrow \infty$. Numerical simulations reveal that for particles sliding down a fluctuating EW surface, $Q_1^* \simeq 0.18$ and $Q_2^* \simeq 0.09$, while $Q_1^* \simeq 0.16$ and $Q_2^* \simeq 0.08$ for the case of driving by a KPZ surface.

A crucial point about the low- m Fourier modes Q_m is that they have broad distributions in the thermodynamic limit. In time, the state shows strong fluctua-

tions in the type and amount of ordering, within a subspace of ordered states. This leads us to call this phenomenon fluctuation-dominated phase ordering (FDPO). It is quite unlike normal ordering in equilibrium phase transitions, where the order parameters, such as the density in a liquid-vapour system or the magnetization in a ferromagnet, have well-defined values at all times in large systems. The order parameter distributions $Prob(\rho)$ and $Prob(m)$ in these cases consist of two delta functions at values characteristic of each phase, i.e. at well-defined densities ρ_v and ρ_l for the fluid, or at magnetizations $\pm m_0$ for the magnet. By contrast, in our system, $Prob(Q_1), Prob(Q_2)$ etc. approach well-defined *broad* distributions in the thermodynamic limit.

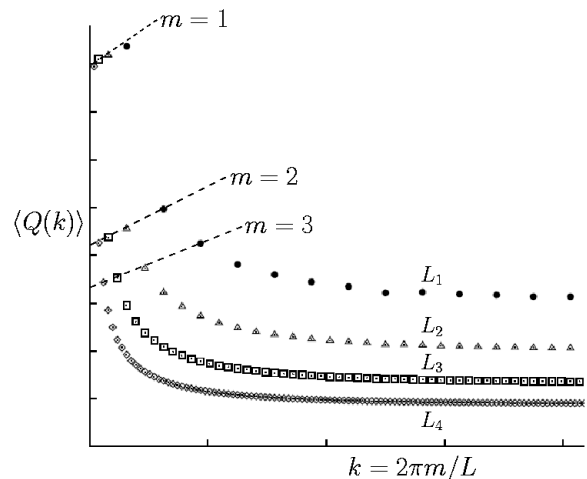


FIG. 3: Plots of $\langle Q(k) \rangle$ versus k are shown for various lattice sizes $L_1 < L_2 < L_3 < L_4$. As L increases, for each fixed low value of m , $Q_m^* = \langle Q_m \rangle$ approaches a limiting value whose significance is discussed in the text.

The values of Q_1^*, Q_2^*, \dots characterize the sort of clustering that occurs in the system. For instance, Q_1 is largest in configurations with a single dense cluster of particles which extends across half the system. On the other hand, Q_2 is largest in configurations with two separated dense clusters. The time series for Q_1 shows that it makes large excursions around its mean value, occasionally reaching high values $\simeq 0.32$, and occasionally low values $\simeq 0$ [10]. It is observed that a dip in the value of Q_1 is usually accompanied by a rise in the value of Q_2 . On the rare occasions when Q_1 and Q_2 are both small, Q_3 picks up. These observations are consistent with strong fluctuations of the ordered structure: a single high-density region is most likely, but it occasionally breaks into two, and more infrequently into three such regions, and so on — never, however, lapsing into a disordered state. FDPO thus involves the system circulating within an attractor of states, each with macroscopic order. In view of the strong anticorrelations between Q_1, Q_2, Q_3, \dots , the best characterization of the state would be through the joint probability distribution $Prob(Q_1, Q_2, \dots)$ but this

remains to be studied systematically.

Evidence for this picture comes from monitoring the length $\ell_{max}(t)$ of the longest cluster in a system of size L [14]. In a disordered state, $\langle \ell_{max} \rangle$ scales as $\log L$. A numerical study of the probability distribution of ℓ_{max} shows that $\langle \ell_{max} \rangle$ increases as L^ϕ where the exponent ϕ ranges from 0.6 to 0.9 for different sorts of surface dynamics, and the full distribution scales with $\langle \ell_{max} \rangle$, implying that the system does not reach a disordered state.

V. SLIDING NONINTERACTING PARTICLES

The related problem of noninteracting particles advected by the Kuramoto-Shivashinsky equation was studied by Bohr and Pikovsky [27], while Chin [28] studied advection by a KPZ field. In both these studies, no noise acts on the particles, in which case clusters do not break up. Besides studying cluster coalescence in time, they monitored the rms displacement of a tagged particle, and found that it grows as $t^{1/z}$ where $z = 3/2$ is the dynamic exponent of the 1- d KPZ model. Drossel and Kardar [26] studied the rms displacement in the presence of noise, and found the same behaviour, while Gopalakrishnan [29] studied the same quantity for driving by an EW surface. Further, Drossel and Kardar studied the density-density correlation function in the cases of particles falling along (advection) and against (anti-advection) the local slope. In the latter case they concluded [25] that correlations show a power law decay ($\sim r^{-\lambda}$) with increasing separation r . We will discuss this further below in the light of our results.

A. Correlation functions and number distribution

Using Model F defined in Section 3 (see Fig. 2b) numerical simulations were performed to monitor the steady state correlation function $C(r, L) = \langle n_i n_{i+r} \rangle_L$ [12, 13]. Both EW and KPZ surfaces were considered in 1- d . Since there is no limit on the particle occupancy n_i at any site, the possibility arises of a much larger degree of clustering. This manifests itself in the fact that the correlation function is a scaling function of r and L . The small-argument singularity of the scaling function shows a divergence:

$$\bar{C}_S(r, L) = L^{-\mu} \bar{Y} \left(\frac{r}{L} \right) \quad (9)$$

with

$$\bar{Y}(y) \sim y^{-\nu} \quad \text{as } y \rightarrow 0 \quad (10)$$

The divergence is strongest for the case of KPZ advection, in which case $\nu = 3/2$ and $\mu = 1/2$. This result agrees with the analytical results of Derrida et al. [31] for a slightly different model, which consists of two particles which slide down slopes, but are not passive in that

they block evolution on the sites at which they reside. In Section 5 below we will see that this form of the correlation function is found also in the adiabatic limit of a quenched interface.

In the case of KPZ anti-advection, simulations show that Eqs. 9 and 10 hold, with $\mu = 0$ and $\nu \simeq 0.31$. The important point is that C is a function of r/L and not r alone. Thus the power law $\sim r^{-\nu}$ has an L -dependent prefactor, unlike the power law behaviour at a critical point. In this respect, our result differs from that of [25, 26].

In the case of EW evolution, Eqs. 9 and 10 are found to hold again, with $\mu = 0$ and $\nu \simeq 2/3$.

The probability that a given site holds n particles $P(n, L)$ also assumes the scaling form

$$P(n, L) \sim L^{-2\delta} f \left(\frac{n}{L^\delta} \right) \quad (11)$$

where

$$f(y) \sim y^{-\gamma} \quad \text{as } y \rightarrow 0. \quad (12)$$

Numerical simulations for the case of KPZ advection yield $\delta \simeq 1$, $\gamma \simeq 1.15$; the numerical results are also consistent with $\gamma = 1$ with logarithmic corrections. For the case of KPZ anti-advection, we find $\delta \simeq 1/3$, $\gamma \simeq 1.7$ while for EW surface dynamics we find $\delta \simeq 0.68$, $\gamma \simeq 1.5$.

Similar results were obtained in two dimensions, using a lattice model similar to that discussed in Section 4.4. The two point correlation function shows a divergence $\sim (r/L)^{-\nu_2}$ with $\nu_2 \simeq 1.4$ for KPZ advection, $\nu_2 \simeq 0.5$ for KPZ antiadvection and $\nu_2 \simeq 0.3$ for EW dynamics.

B. Adiabatic limit: the Sinai model

Consider the adiabatic limit in which the surface is absolutely still, while particles evolve by performing biased random walks, with the bias on each link set by the slope of the quenched surface. The problem then reduces to the Sinai problem of noninteracting random walkers moving in a random medium which itself is defined as the trajectory of a random walk. Since the surface is static, the walkers have infinite time to explore the landscape, and eventually reach an equilibrium state defined by a temperature $T = V \ln(p_2/q_2)$. Thus $\langle n_i n_{i+r} \rangle$ is given by $\exp[-\beta(h_i + h_{i+r})]/Z(\{h_k\})$ for a particular configuration of heights $\{h_k\}$. We need to further average $\langle n_i n_{i+r} \rangle$ over quenched disordered landscapes $\{h_k\}$, a computation that was carried out by Comtet and Texier [32]. Taking the scaling limit $r \rightarrow \infty$, $L \rightarrow \infty$ with r/L held fixed, we find that $\langle n_i n_{i+r} \rangle$ has the scaling form of Eq. 2, with

$$\bar{Y}(y) = (2n\beta^2 L)^{-1/2} [y(1-y)]^{-3/2} \quad (13)$$

where $y = r/L$.

Further, the probability $P(n, L)$ that a site holds n particles can be calculated as well [14]. It has the scaling form $P(n, L) = 4/\beta^4 L^2 \bar{X}(x)$ with $x = 2n/\beta^2 L$ and

$$\bar{X}(x) = e^{-x} K_0(x)/x \quad (14)$$

where $K_0(x)$ is a Bessel function of imaginary argument.

Surprisingly, we find very good agreement between these results for an equilibrium state of particles in a disordered environment, and those for the strongly nonequilibrium situation of KPZ advection discussed in Section 5.1. The correlation scaling function $\bar{Y}(y)$ in Eq. 13 fits the numerical data closely, if we set $\beta \simeq 4$. On the other hand, the scaling function $\bar{X}(x)$ in Eq. 14 describes the probability density data for the nonequilibrium system provided we set $\beta \simeq 2.3$. Although the driving force is different in the two cases – being the temperature in the equilibrium disordered system, and surface fluctuations in the nonequilibrium system – it allows particles to explore the terrain in both cases, and reach states with similar though not identical characteristics.

VI. CONCLUSION

The principal conclusion of this paper is that in a class of inhomogeneously clustered states, the density-density correlation function is a function not of the separation alone, but rather the ratio of separation to system size (in steady state) or separation to a growing length scale (in the coarsening regime). This scaling behaviour is reminiscent of phase ordered states, but unlike conventional phase ordering, the scaling function here is singular at small argument. In a system of particles with mutual exclusion sliding down a fluctuating surface, the singularity in question is a cusp, while for noninteracting particles which can cluster more strongly, the singularity is a divergence. There is a degree of universality in the exponents which characterize the singularities in scaling functions: they remain unchanged under variation of parameters in the model, for instance, the ratio of rates of particle movement and surface fluctuations, though they

do depend on the symmetry and dimension of the driving field.

The other hallmark of the states under consideration here is the occurrence of giant fluctuations, characterized by the standard deviation of the order parameter growing proportionally to the mean, as a consequence of a probability distribution that remains broad in the thermodynamic limit.

It is to be emphasized that the fluctuation-dominated states under discussion here are quite different from critical states. In critical states, correlation functions decay as power-laws, and size effects appear as corrections. By contrast, correlation functions in our case are, even in leading order, functions of the separation scaled by system size. Moreover, fluctuations at critical points lead to probability distributions that narrow down in the thermodynamic limit, unlike the giant fluctuations present here.

The occurrence of singular scaling functions is not confined to the models studied in this paper. A study of nonequilibrium nematic states shows that the density-density correlation function shows similar scaling properties, with a cuspy behaviour of the scaling function [15]. Further, a recent study of inelastically colliding particles for which the coefficient of restitution depends on the relative velocity of approach of two particles, shows similar non-Porod scaling properties [16].

Finally, there is an intriguing connection between the properties of the strongly nonequilibrium system under study and an equilibrium system with quenched disorder which is obtained by freezing the fluctuating potential. We have presented evidence for the similarities of several properties, but a deeper understanding of the connection remains an open question.

Acknowledgements: I would like to thank Dibyendu Das, Manoj Gopalakrishnan, Apoorva Nagar, Sakuntala Chatterjee and Satya Majumdar for enjoyable collaborations on the passive particle problems discussed in this paper. I am grateful to Shamik Gupta for useful comments on the manuscript.

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